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THE LOGARITHMIC POTENTIAL

DISCONTINUOUS DIRICHLET AND NEUMANN PROBLEMS

BY

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TO

VITO VOLTERRA

PREFACE

This small treatise is an outgrowth of a study of Stieltjes integrals and potential theory which the author published in the 1920 volume of the Rice Institute Pamphlet, and a needed revision and development of the last part of that essay in the direction indicated by three notes which appeared in 1923, in the Comptes rendus des séances de l'Académie des Sciences. Two of these were written in conjunction with my colleague, Professor H. E. Bray. The work gives a unified treatment of the basis of the theory of Laplace's equation in two dimensions, suitable, it is hoped, for graduate students of a moderate degree of advancement, and is intended to be of service in the development of the theory of partial differential equations of elliptic type. These developments are generating a compound of two of the most important elements of modern analysis—the concepts of Lebesgue on the one hand, and of Volterra on the other.

An earlier form of part of the treatise was given in lectures at the Rice Institute in the academic year 1924–25, in connection with a course in the theory of functions of a real variable, and at the University of Chicago during the Summer Quarter of 1925. Chapter VII furnished the substance of an invited discourse at the meeting of the Southwestern Section of the American Mathematical Society in November, 1926.

The author is much indebted to Professor O. D. Kellogg, who has seen a large portion of the manuscript, and aided with kindly criticism, to Professor Bray, who has read the proof sheets, and, finally, to the American Mathematical Society, through whose generosity the publication is possible.

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GRIFFITH C. EVANS.

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CHAPTER I

PRELIMINARY CONCEPTS
STIELTJES INTEGRALS AND FOURIER SERIES

1. Functions of limited variation. In the chapters which follow, the Dirichlet and Neumann problems are recast under general points of view which derive, as did earlier formulations of the problems, from physical considerations. These problems are to be discussed in relation to conformal transformations and the most general distributions of matter, simply or in doublets, on the boundary of a general simply or finitely connected open region bounded by circles. In this way new classes of boundary conditions arise, and the appropriate boundary value problems may be solved. Thus for the old problems some new results, as well as some familiar ones, appear as special cases.

The principal instrument for the investigation of harmonic functions, from this point of view, will be the *Stieltjes integral.* This integral refers fundamentally to functions *of limited variation.* For the convenience of the reader some important properties of the relation between these two concepts will be briefly summarized, and one or two theorems obtained.

Let $f(x)$ be defined for every x in the closed interval (a, b), and let this interval be divided into a finite number n of subintervals by points $x_1, x_2, \cdots, x_{n-1}$, writing $x_0 = a$, $x_n = b$ for convenience. If the quantity

$$\sum_{0}^{n-1}{}_i |f(x_{i+1}) - f(x_i)|, \quad x_0 < x_1 < x_2 \cdots < x_n,$$

is bounded, $\leqq N$, for all such positions of the x_i in (a, b) and all n, the function is said to be *of limited variation* in (a, b). The least value of N which will satisfy this condition is said to be the *total variation* of $f(x)$ in (a, b), and is designated by T.

In any closed interval (a, x), $a < x \leqq b$, let us choose a set of non-overlapping subintervals (x_1', x_1''), (x_2', x_2''), $\cdots$,

finite in number, which add together into the whole or less than the whole of (a, x). The upper bound, for all such choices, of the quantity

$$\sum_k (f(x_k'') - f(x_k'))$$

exists, and we designate it by $\varphi(x)$ or φ_{ax}; that is to say, $\varphi(x)$ is the smallest number which is not exceeded by any possible value of the given sum. Let us definitely admit a single point as a special case of an interval, namely one whose end points coincide. We see then directly that $\varphi(x) \geqq 0$, and again that $\varphi(x)$ is a non-decreasing function of x. It is called the *positive variation function* of $f(x)$.

Exercise. Show that $\varphi(x)$ has the same value and the same properties even if we do not admit intervals of zero length. These properties may be demonstrated by proving that given ε arbitrarily small we can find in any interval (x_1, x_2) some subinterval (x', x'') for which $|f(x'') - f(x')| < \varepsilon$.

In order to complete the definition of $\varphi(x)$ we must assign it a value at $x = a$; we assign it there the value 0.

It is interesting to note that if X is some value intermediate between a and x we have

$$\varphi_{ax} = \varphi_{aX} + \varphi_{Xx}$$

or

$$\varphi(x) = \varphi(X) + \varphi_{Xx}.$$

For we can form a sum for the interval (a, x) as near to φ_{ax} as we desire; if X is an interior point of a subinterval the sum is not changed if we insert X as the end of one interval and the beginning of the next; thus in any case we can form partial sums which relate to φ_{aX} and φ_{Xx} respectively, yet cannot exceed them. Hence $\varphi_{ax} \leqq \varphi_{aX} + \varphi_{Xx}$. Also we can form partial sums for (a, X) and (X, x) as near respectively to φ_{aX} and φ_{Xx} as we desire; the sum total will be $\leqq \varphi_{ax}$. Hence $\varphi_{ax} \geqq \varphi_{aX} + \varphi_{Xx}$, from which the conclusion follows.

We define another function of x, which is a non-increasing function:

$$\psi(x) = f(x) - f(a) - \varphi(x).$$

In fact, if x_1, x_2 are two values of x, with $x_2 > x_1$, we have

$$\psi(x_2) - \psi(x_1) = f(x_2) - f(x_1) - \{\varphi(x_2) - \varphi(x_1)\}.$$

But $\varphi(x_2) - \varphi(x_1) = \varphi_{x_1 x_2} \geqq f(x_2) - f(x_1)$. Hence $\psi(x_2) - \psi(x_1) \leqq 0$, and $\psi(x)$ is a non-increasing function of x.

The function $\psi(x)$ is called the *negative variation function* of $f(x)$. Moreover by the definition of $\psi(x)$ we have

$$f(x) = f(a) + \varphi(x) + \psi(x).$$

In other words, a function of limited variation in the closed interval (a, b) can be written as the difference of two non-decreasing functions of x. The converse of this statement is obviously true.

The function $t(x) = \varphi(x) - \psi(x)$ is again a non-decreasing function of x, being the sum of two such functions, and is called the *total variation function* of $f(x)$.

EXERCISE. Show that definitions of $\psi(x)$ and $t(x)$ analogous to that of $\varphi(x)$ may be given, and that $t(b) = T$.

A non-decreasing function approaches a limiting value as x approaches a value α from the left, and also as x approaches α from the right. Hence the same property holds for the difference between two such functions, that is, for any function of limited variation. The two values, the limit from the left and the limit from the right, need not be equal, or equal to the value $f(\alpha)$, since $f(x)$ need not be continuous at α. It is therefore convenient to have symbols for these values and write $f(\alpha + 0)$ for the quantity $\lim\limits_{\substack{x=\alpha \\ x>\alpha}} f(x)$, and $f(\alpha - 0)$ for the quantity $\lim\limits_{\substack{x=\alpha \\ x<\alpha}} f(x)$, respectively. If $\alpha = 0$, the symbols $f(0+)$ and $f(0-)$ are used. The statement we have just made then amounts to saying that if $f(x)$ is of limited variation, $f(x+0)$ and $f(x-0)$ exist if $a < x < b$; and also $f(a+0)$ and $f(b-0)$ exist.

An important property of a function of limited variation is that the aggregate of its points of discontinuity must be denumerable. To prove this fact it is sufficient to consider

a non-decreasing function, since the discontinuities of a function of limited variation will also be discontinuities of its total variation function. But for such a function the number of points x where $f(x+0)-f(x)$ or $f(x)-f(x-0)$ is $\geqq T/2$ is finite; also the number of points such that these jumps are $< T/2$ but $\geqq T/4$; also the number of points for which these jumps are $< T/4$ but $\geqq T/8$, etc. In such a classification however every point of discontinuity is ultimately included, since a point where both $f(x+0)-f(x)$ and $f(x)-f(x-0)$ are both zero is a point of continuity.

2. Continuation. We have discussed the function of limited variation with respect to a closed interval (a, b). It is convenient however to be able to consider the same sort of situation with respect to an open interval. That is we say that $f(x)$ is of limited variation if

$$\sum |f(x_k'')-f(x_k')| \leqq N,$$

N being some constant, no matter how the subintervals (x_k', x_k'') are chosen in the open interval (a, b). We define the positive variation function as before; it is again a non-decreasing function of x, with $\varphi(a+0)=0$; moreover $\varphi(x)=\varphi(X)+\varphi_{Xx}$ if $a<X<x<b$. Further let $\psi_1(x)=f(x)-\varphi(x)$; this is a non-increasing function. Hence $\psi_1(a+0)$ exists. We therefore define the negative variation function as

$$\psi(x) = \psi_1(x)-\psi_1(a+0),$$

so that $\psi(a+0)=0$. We define the total variation function as

$$t(x) = \varphi(x)-\psi(x),$$

and we have also $t(a+0)=0$. Whether or not for completeness we define $\varphi(a)$, $\psi(a)$ and $t(a)$ as 0 is immaterial, since a is outside the open interval. Finally, we have

$$f(x) = \varphi(x)+\psi(x)+\psi_1(a+0),$$

so that $f(a+0)=\psi_1(a+0)$. The quantities $f(a+0)$ and $f(b-0)$ are thus seen to be determinate, although $f(a)$ and $f(b)$ are not defined. The quantity $t(b-0)-t(a+0)$ is seen to